Jon Michael Smith on T-Integration

Trade secrets in numerical analysis

Introduction

Students from my American Society of Mechanical Engineers (ASME) seminars on Math Modeling for Simulation and Controls suggested that I put together Web pages on the most useful numerical methods I have found in 30 years of developing numerical methods for real-world applications. So here they are. This is the first of a set of algorithms and formulae that I use frequently when tuning software for control systems and for both scaled-time simulation and for real-time simulation. Those who visit this site should benefit some day from having these methods available on the Web. The methods will be published on an irregular schedule, so check back to this Web page from time to time to see what else has been included.

Trade Secrets

These methods are my trade secrets. They give me a competitive edge over other systems and software engineers when it comes to solving simulation and computer controlled system problems that I encounter in my career and in my small business consulting practice. Many are numerical methods for solving digital simulation and controls problems that others did not know or had not developed. Some were developed for engineering and some for business. Some are published in the papers and books I wrote, the most detailed of which is *Mathematical Modeling and Digital Simulation for Engineers and Scientists 2ed Edition*; John Wiley and Sons Publishers way back in 1988. Some are taught in my ASME seminars in the early 90's. Most have never been revealed before. I hope this project will be useful to engineers, my sons included, who want to be competitive and ***prosper*** through engineering. For sure, these methods have, and continue to be, useful to me in that important regard. The first, and most useful, and somewhat of a trade secret known only to my ASME students, and readers of my books is a method I call T-Integration.

T-Integration

T-Integration is a simple numerical integration formula that is fast, accurate, numerically stable and can be used for both real-time and scaled-time and quadrature applications is:

$$\mathbf{X_n = X_{n-1} + TG\,[\,P(dX/dt\,)_n + (1\text{-}P)(dX/dt\,)_{n-1}\,]\,.}$$

Here X is the integral, dX/dt is the integrand, P and G are "gain" and "phase" tuning parameters (taken from information theory in the frequency domain), and n refers to the number of the iteration being evaluated and T is the integration step size (taken from approximation theory in the time domain). The notion here is that algorithms developed with both the time domain and the frequency domain considerations are better than algorithms developed from either one alone. Controlling and Simulating modern systems that include both analog processes being controlled or simulated with digital computers simply require both points of view be integrated into algorithms being used for control or simulation.

Background - Solving the linear problem

This integrator has been used in real-world applications including simulators used in the Apollo program, commercial aircraft training simulators produced by the old Conductron Company, the early DC-10 automatic landing system performance and failure assessment monitors (PAFAM), Lockheed missile simulators and other applications that my colleagues from the ASME seminars have found useful. It was developed to solve numerical instability and accuracy problems encountered in simulating entry of the Apollo command module. At that time my friends Elliot Pyron, Brian Schoonmaker and Maurice Fowler, and I were working on the problem of correctly simulating linear continuous control systems on a digital computer. At that time Elliot, Brian and I were working for the McDonnell Douglas Houston Operations in Houston Texas under the direction of Chuck Jacobson. Maurice was at IBM in Palo Alto California. Later he joined us in Houston to be a main contributor to simulator development on the Apollo program and on the DC-10 PAFAM development.

We had worked on this problem for a year or more and finally came up with numerical methods that were intrinsically stable and accurate and could not be made to go unstable and faithfully simulated the dynamics of real systems. Maurice was really the first to solve this problem when he was at IBM. Elliott, Brian and I had taken a different tack and had come up with identical answers, but from a different point of view. We were all happy to have the methods because they were, and still are, the competitive edge in linear systems simulation and linear control systems software design.

Solving the nonlinear problem

Having found that method (to be covered later), Elliott and I backed out the numerical integrator that was the basis for our success with linear systems and found to our surprise that it was a simple zero-order numerical integrator with TWO parameters that could be set analytically or empirically. In this sense the integrator could be TUNED to the problem it was solving, whether linear or nonlinear; thus the name T-integration. At that point we developed a number of T-integrators of higher order and published the work in a paper entitled *"Recent Developments in Numerical Integration"* in the ASME Journal of Dynamic Systems, Measurement and Control.

During that period, it became clear that all classical zero-order numerical integrators were special cases of the zero-order T-integrator. A further finding was that the zero-order T-Integrator could be reduced to an Euler Integrator with Lagrange's interpolator/extrapolator. This then was the final breakthrough that leads to tuning numerical integrators of all orders and types by many mathematicians for application in both linear and nonlinear simulations and control systems. The idea was to simply phase shift the integrand and adjust its gain to match whatever check-case or mathematical criteria was being employed. We even found ways to do this analytically (more later). This technology became the basis for the formation of a business, JMSA (J.M. Smith and Associates), seminars, and consulting contracts and these Web pages. Most recently, I have been using the T-Integrators in simulations of chaotic phenomena; more on that important problem in another trade secret web page, later.

Zero Order T-Integrator details

Back to T-integration... This simple first order integrator can be tuned to the system of equations it is being used to solve, whether linear or nonlinear. It was developed during the Apollo program to be used in real-time training simulators for training Astronauts on flight guidance and control procedures. By appropriate selection of *P* and *G* this integrator can be made to **exactly** integrate the homogeneous solution to linear differential equations. This IS NOT POSSIBLE with classical integrators such as the Runge-Kutta series or the Predictor-Corrector series or any other methods in the literature, even today. To do so it is only necessary to derive a simulating difference equation by numerical integrator substitution into the differential equation and the match the poles and zeros of the resulting difference equation with the poles and zeros of the differential equation. This is done by algebraically solving for *P* and *G* in terms of the poles and zeros of the differential equations. The resulting numerical integration of the inhomogeneous equation will not be exact but will be accurate (see reference below on the mean value theorem of integral calculus) and **intrinsically stable**, provided, of course, if the system of differential equations is stable. Knowing that the numerical integrator cannot become numerically unstable makes this integrator ideal for use in real-time controls systems and simulators.

For use with nonlinear systems, the values for *P* and *G* can be determined empirically to be accurate and stable for ranges of the dynamic variables in the problem being simulated or controlled. For example, when use in flight simulators, the values for *P* and *G* can be determined by simulating aircraft with various combinations of landing gear position, wing flap position, wing spoiler position, mach number, engine out configurations etc., and adjusting *P* and *G* to match known check cases. Then, the *P's* and *G 's* are changed in the numerical integrator based on the position of the landing gear control, the flap control, Mach number, positions of the throttles etc. Doing so tunes the numerical integrator to the flight dynamics and aerodynamic configuration of the aircraft being simulated.

As I mentioned earlier, an important mathematical aspect of the T-Integrator is that for *G* = 1 and varying *P* from 0 to 2 in ratios of integers results in most of the useful classical first order integrators. For example, when

- *G* = 1 and *P* = 0, the T-Integrator becomes the Euler integrator,
- *G* = 1 and *P* = 1/2, the T-Integrator becomes the Trapezoidal integrator,
- *G* = 1 and *P* = 1 the integrator becomes the Rectangular integrator,
- *G* = 1 and *P* = 3/2 the integrator becomes the Adams-Bashforth Corrector (ABC).

This is important because it gives some clients the confidence that the T-Integrator contains old friends that are will documented in mathematical literature, both their applications and limitations. Interestingly, this is important for real time simulations because the ABC integrator is still often used as a single step integrator for real-time simulations (following MIT's classical papers on numerical integrators for real time simulation done in the sixties). The list of classical integrators that are special cases of the T-integrator is long. **What is more important, however, is that the T-integrator can take on a double infinity of values (G and P) in-between the classical integrators**. In this sense the integrator can be tuned precisely to any problem being solved, system being controlled or system being simulated.

Wikipedia Summary

T-Integration can be tuned to the problem it is being used to solve. T-Integration is based on information theory, not approximation theory. T-Integration has very simple frequency domain adjusting parameters: A phase adjusting parameter and a gain adjusting parameter. Interestingly, for open-loop problems, setting the gain and varying the phase produces ALL classical first order numerical integrators and an infinity new integrators heretofore unknown. For closed loop applications the T-Integrator produces an unlimited number of non-classical integrators that produce exact numerical integration of linear systems and near exact integration of nonlinear systems.

For information systems applications (computer, control and communication and simulation) the simple first order T-Integrator out performs all numerical integrators based on classical approximation theory. Simulating aircraft motion for various aircraft configurations:

- gear up
- gear down,
- flaps up,
- flaps down,
- left engine out,
- right engine out,
- stab-augmentation on,
- stab-augmentation off
- etc

and various flight dynamic conditions

- high mach,
- low mach,
- high altitude,
- low altitude,
- take-off,
- cruise,
- landing,

becomes a simple matter of tuning the T-Integrator to the matrix of flight configurations and flight conditions being simulated. In this sense the T-Integrator adapts to the problem it is trying to simulate.

From a controls system and real-time simulation point of view, first order numerical integrators are important for digital computer simulation of linear continuous systems because they do not introduce additional poles or zeros in the digital equivalent of linear continuous systems being simulated. Similarly, for nonlinear continuous systems, they do not introduce additional terms in the Jacobian of the digital equivalent of the nonlinear continuous system being simulated.

References

This type of numerical integrator is called a T-Integrator because it can be tuned to the problem it is trying to solve. For further information please contact me at my e-mail address below or read the chapter on **Modern Numerical Integration Methods** in the book "Mathematical Modeling and Digital Simulation, 2ed Edition", 1988, ISBN 0-471-08599-5, *John Wiley and Sons Publishers*. Other early, but important references that have had the test of time, and critical review include;

- Eric.W.Weisstein, "T-Integration Citation," *CRC Concise Encyclopedia of Mathematics, Second Edition,"* pp.2986, 2003.
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- Maurice Fowler, "A New Numerical Method for Simulation," *Simulation,* Vol. 6, No. 2, pp. 90-92, February 1976; Vol. 8, pp. 308-310, June 1967.
- http://mathworld.wolfram.com/T-Integration.html

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If you have a particularly difficult time integrating a system of differential equations then call me. It does not matter how stiff or loose the system or what order the system or how nonlinear the system; they all get my attention. Numerical analysis for dynamic systems control or simulations is a large part of my life's work. Helping solve such problems provide you with solutions and provide me with the challenge of tuning and adapting numerical methods (including the T-Integrator) to practical applications.

If you are having a difficult time simulating a digitally controlled analog process and you need results, not research papers, then call me. We can work out a consulting and/or training arrangement that will be both helpful to solve your problem and instructive for the engineers and software developers on your staff. As I said before, it's my life's work and I enjoy seeing it applied in useful ways. I can be reached at jmsa@jonmsmith.com.

Thank you for visiting my **trade secrets** website. 18 April 2007.